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Address of the editorial office
Goce Delcev University – Stip
Faculty of philology
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CONTENT

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APPLICATION OF SECOND - ORDER NONHOMOGENEOUS DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS IN SERIAL *RL* PARALLEL *C* CIRCUIT

Goce Stefanov, Jasmina Veta Buralieva, Maja Kukuseva Paneva, Biljana Citkuseva Dimitrovska

Abstract. In this paper, a solution of second order differential equation and its real usage in analyzing electronic circuits are represented. First, a mathematical analysis of the differential equation is made, and then its solution to a particular case is used to analyze serial *RL* parallel *C* resonant circuit. The results of simulations about the currents and voltages in the circuit are also represented.

1. Introduction

The basic electronic circuits are composed of passive and active elements such as resistors, inductors and capacitors. The currents and voltages in the circuit in a time domain are described by integral- differential equations. The type and order of the equations depend on the number of inductors and capacitors in the circuit. If the circuit is made of resistors only, the currents and voltages are described by algebraic equations. If there is one inductance (or capacitor) in the circuit, the currents and voltages are described by a first-order integral-differential equation. If the number of inductors and capacitors is n, then the currents and voltages will be described by an integral differential equation of n^{th} order, [1], [2], [3], [4].

In electrical engineering, there are techniques for solving these integral-differential equations by which the time domain equations are transformed into a frequency domain. These are the methods of complex analysis, Fourier transform and Laplace transformation, [5], [6], [7]. Mainly by applying these methods a satisfactory solution of the equations is obtained at a certain interval. Solving integral- differential equations in time- domain is more complex, but the results give a picture of the actual state of the circuit. Therefore, this paper deals with solving equations in time domain, [8].

This paper is organized as follows: In Section 2, a second- order nonhomogeneous linear differential equation with constant coefficients is considered and its solution is represented. In Section 3, a serial *RL* parallel *C* circuit is analyzed and the solution of the second- order differential equation that describes this circuit is represented. The simulation of the circuit and the obtained waveforms for the current and voltage under various conditions are represented in Section 4. Section 5 concludes this paper.

2. Second - Order Nonhomogeneous Linear Differential Equation with constant coefficients

In this section a second - order nonhomogeneous linear differential equation with constant coefficients is considered and its solution is represented. This kind of differential equations is frequently used in engineering for solving practical problems. One can find more about the second - order nonhomogeneous linear differential equations with constant coefficients and their application in [5], [6], [7].

If,

$$a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = G(t), \quad a \neq 0 \tag{2.1}$$

is second - order nonhomogeneous linear differential equation where a, b and c are constants, and $G(t)$ is a continuous function. The related homogeneous equation

$$a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = 0, \quad a \neq 0 \tag{2.2}$$

is usually called the complementary equation and it plays an important role in the solution of the original nonhomogeneous equation (2.1).

The general solution of the nonhomogeneous linear differential equation (2.1) can be written as:

$$y(t) = y_p(t) + y_c(t) \tag{2.3}$$

where $y_p(t)$ is a particular solution of the equation (2.1) and $y_c(t)$ is the general solution of the complementary equation (2.2).

In order to solve the homogeneous equation (2.2), the characteristic equation $as^2 + bs + c = 0, \ \ a \neq 0$ is formed with roots s_1 and s_2. Then, if s_1 and s_2 are real and different, the solution of (2.2) has the form:

$$y_c(t) = Ae^{s_1 t} + Be^{s_2 t}; \tag{2.4}$$

if $s_1 = s_2$ then

$$y_c(t) = (At + B)e^{s_1 t}; \tag{2.5}$$

and if $s_1 = \varphi + i\psi$ and $s_2 = \varphi - i\psi$, i.e., complex conjugate, then

$$y_c(t) = e^{\varphi t}(C\cos\psi\, t + D\sin\psi\, t). \tag{2.6}$$

One can find the particular solution $y_p(t)$ of the differential equation (2.1) by two methods, i.e. the method of undetermined coefficients which is straightforward but works only for a restricted class of functions, or by the method of variation of parameters which works for every function but is usually more difficult to apply in practice.

The method of undetermined coefficients means that the particular solution has the same form as the continuous function $G(t)$, i.e., if $G(t) = e^{kt}P(t)$ where $P(t)$ is a polynomial of degree n and k is a constant, then $y_p(t) = e^{kt}Q(t)$ where $Q(t)$ is an n^{th} degree polynomial (whose coefficients are determined by substituting in the differential equation); if $G(t) = e^{kt}P(t)\cos r\, t$ or $G(t) = e^{kt}P(t)\sin r\, t$ where $P(t)$ is an n^{th} degree polynomial, then $y_p(t) = e^{kt}Q(t)\cos r\, t + e^{kt}R(t)\sin r\, t$, where $Q(t)$ and $R(t)$ are n^{th} degree polynomials. Let us note that if any term of $y_p(t)$ is a solution of the complementary equation (2.2) and therefore can't be a solution of the nonhomogeneous equation, then one needs to multiply $y_p(t)$ by t (or by t^2 if it is necessary).

For the Method of Variation of Parameters, the solution of the homogeneous equation (2.2) has the form $y(t) = c_1y_1(t) + c_2y_2(t)$ where $y_1(t)$ and $y_2(t)$ are linearly independent solutions, and c_1 and c_2 are constants. If the constants c_1 and c_2 are replaced by arbitrary functions $u_1(t)$ and $u_2(t)$, then the particular solution of the nonhomogeneous equation (2.1) has the form $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where $u_1(t)$ and $u_2(t)$ can be found as an integral of the solution of the system $u_1'(t)y_1(t) + u'_2(t)y_2(t) = 0$ and $a(u_1'(t)y_1'(t) + u_2'(t)y_2'(t)) = G(t)$.

3. Serial *RL* parallel *C* resonant circuit

A serial *RL* parallel *C* resonant circuit is shown in Figure 1. This is a base circuit in power electronic and it has application in power converters such as DC/AC/DC resonant converter and induction heating furnaces.

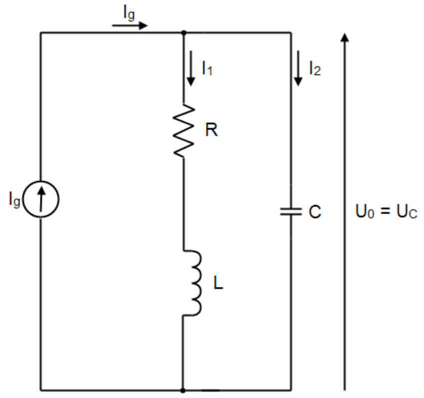


Figure 1. *Serial RL parallel C resonant circuit*

The current generator i_g is a square generator that supplies this circuit. This generator has the switching frequency f_s (period T_s), and the maximal current amplitude $\pm k$. The resistance of the resistor is R=0.4 Ω, inductance of the inductor is $L = 1.2$ mH, the capacitance of the capacitor is C=1.1 μF. The current through the inductor is denoted with I_1, I_2 is the capacitance current, while U_c is the capacitor voltage, which is the output circuit voltage.
According to Kirchhoff current law, it is obtained:

$$i_g(t) = i_1(t) + i_2(t) \tag{3.1}$$

From the Kirchhoff voltage law for voltages of the second brunch, it is obtained:

$$u_c(t) = Ri_1(t) + L\frac{di_1(t)}{dt} \tag{3.2}$$

The current through the first branch, according to Kirchhooff current law, is:

$$i_1(t) = i_g(t) - i_2(t) \tag{3.3}$$

With substitution of equation (3.3) in equation (3.2) it is obtained:

$$u_c(t) = Ri_g(t) - Ri_2(t) + L\frac{di_g(t)}{dt} - L\frac{di_2(t)}{dt} \tag{3.4}$$

The current through the second branch, i.e. the current through the capacitor is:

$$i_2(t) = i_c(t) = C\frac{du_c(t)}{dt} \tag{3.5}$$

The substitution of equation (3.5) in equation (3.4) leads to:

$$u_c(t) = Ri_g(t) - RC\frac{du_c(t)}{dt} + L\frac{di_g(t)}{dt} - LC\frac{d^2u_c(t)}{dt^2} \tag{3.6}$$

$$LC\frac{d^2u_c(t)}{dt^2} + RC\frac{du_c(t)}{dt} + u_c(t) = L\frac{di_g(t)}{dt} + Ri_g(t) \tag{3.7}$$

Dividing equation (3.7) by LC, the second - order differential equation is obtained:

$$\frac{d^2u_c(t)}{dt^2} + \frac{R}{L}\frac{du_c(t)}{dt} + \frac{1}{LC}u_c(t) = \frac{1}{C}\frac{di_g(t)}{dt} + \frac{R}{LC}i_g(t) \tag{3.8}$$

The equation (3.8) describes the circuit from Figure 1. The solution of the differential equation depends on the steady state condition.

In this paper it is considered a special case when $i_g(t)$ is a square current generator, i.e. $i_g(t) = \begin{cases} k, & 0 \le t \le \frac{T_s}{2} \\ -k, & \frac{T_s}{2} \le t \le T_s \end{cases}$, where T_s is period. Then the equation (3.8) has the form:

$$\frac{d^2 u_c(t)}{dt^2} + \frac{R}{L}\frac{du_c(t)}{dt} + \frac{1}{LC}u_c(t) = \frac{R}{LC}i_g(t). \tag{3.9}$$

The solution of the differential equation (3.9) is $u_c(t) = u_{c_c}(t) + u_{c_p}(t)$ where $u_{c_p}(t)$ is the particular solution of (3.9) and $u_{c_c}(t)$ is the general solution of the complementary equation of (3.9).

According to Section 2, the particular solution of the equation (3.9) is a constant, which means that $\frac{du_{c_p}(t)}{dt} = 0$ and $\frac{d^2 u_{c_p}(t)}{dt^2} = 0$. Then it turns out that (3.9) has the form:

$$\frac{1}{LC}u_{c_p}(t) = \frac{R}{LC}i_g(t)$$

and $u_{c_p}(t)$ is really constant and has the form:

$$u_{c_p} = Ri_g(t) = \begin{cases} Rk, & 0 \le t \le \frac{T_s}{2} \\ -Rk, & \frac{T_s}{2} \le t \le T_s \end{cases}. \tag{3.10}$$

In order to find the solution of the complementary equation

$$\frac{d^2 u_c(t)}{dt^2} + \frac{R}{L}\frac{du_c(t)}{dt} + \frac{1}{LC}u_c(t) = 0, \tag{3.11}$$

The corresponding characteristic of equation (3.11) is constructed:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{3.12}$$

such that the solutions can be written as:

$$s_{1/2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

with respect to the parameters R, C and L.

Three cases can be considered:

1. if $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$, the roots of the characteristic equation (3.12) are equal, i.e., $s_1 = s_2 = -\frac{R}{2L}$, then the solution of the homogeneous equation (3.11) has the form
$$u_{c_c}(t) = (At + B)e^{-\frac{R}{2L}t}; \tag{3.13}$$
2. if $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$, the roots of the characteristic equation (3.12) are real and different, i.e. $s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ and $s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$, and the solution of the homogeneous equation (3.11) has the form:
$$u_{c_c}(t) = Ae^{\left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t} + Be^{\left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t}; \tag{3.14}$$

3. when $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$, the roots of the characteristic equation (3.12) are complex conjugate, i.e. $s_1 = -\frac{R}{2L} + i\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$ and $s_2 = -\frac{R}{2L} - i\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$. Then the solution of the homogeneous equation (3.11) has the form

$$u_{c_c}(t) = e^{-\frac{R}{2L}t}\left(C\,cos\left(\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right)t + D\,sin\left(\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right)t)\right) \quad (3.15)$$

Then, by (3.13), (3.14), (3.15) and (3.10), the solution of the differential equation (3.8) can be written as:

$$u_c(t) = (At + B)e^{-\frac{R}{2L}t} + Ri_g(t),$$

$$u_c(t) = Ae^{\left(-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right)t} + Be^{\left(-\frac{R}{2C}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right)t} + Ri_g(t),$$

$$u_c(t) = e^{-\frac{R}{2L}t}\left(C\,cos\left(\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right)t + D\,sin\left(\sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right)t)\right) + Ri_g(t),$$

(3.16)

respectively.

In electronics, the values of the elements that satisfy the third case (solution) of the differential equation (3.11) with complex conjugate solution are mostly used. This case is also known as a pseudo- periodic resonance. The damping circle frequency is $\omega_d = \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$ with $\alpha = \frac{R}{2L}$ as a damping factor. Then the solution of the differential equation (3.9) is

$$u_c(t) = e^{-\alpha t}(C\,cos(\omega_d t) + D\,sin(\omega_d t)) + Ri_g(t) \quad (3.16)$$

In the pseudo- periodic case, the equation (3.16) can be written as:

$$u_c(t) = e^{-\alpha t}K\,sin(\,\omega_d t + \varphi) \quad (3.17)$$

In (3.17) the parameter K is the maximal value of the output circuit voltage, and φ is the phase angle between the current and the voltage in the circuit. The power that the source develops to the consumer depends on this angle.

4. Results from simulations and discussion

Figure 2 shows a serial *RL* parallel *C* circuit for simulations. The simulations are made in the PowerSim program, [9].

For the values of the elements defined above, the damping frequency is $f_d = \omega_d/(2\cdot\pi)$= 4367 Hz and it is close to the resonant frequency $f_o = 1/(LC)^{1/2}$. At this frequency phase, the angle is close to zero and the consumer receives most power from the source. Figure 3a shows the waveforms on the voltage and current in the circuit when the switching frequency is the same with the damping frequency, $f_s = f_d$ = 4367 Hz. Figure 3b shows the waveforms of the voltage and current in the circuit for a switching frequency smaller than the damping frequency, f_s = 4350 Hz < f_d. In Figure 3c the waveforms of the voltage and current in the circuit for switching frequency bigger than the damping frequency, f_s = 4395 Hz > f_d are shown.

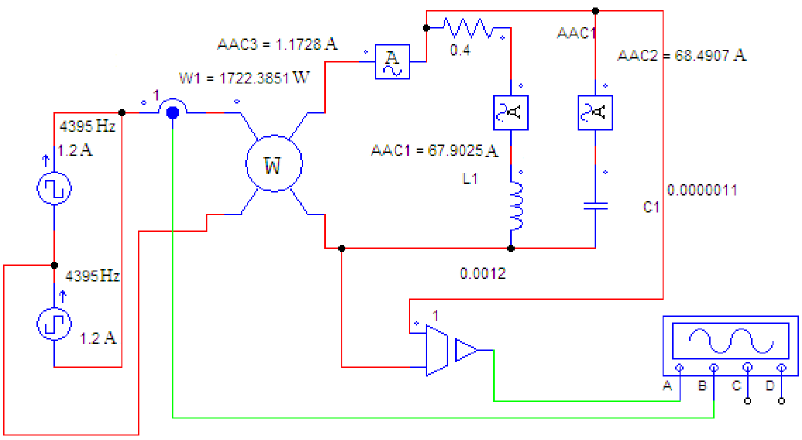
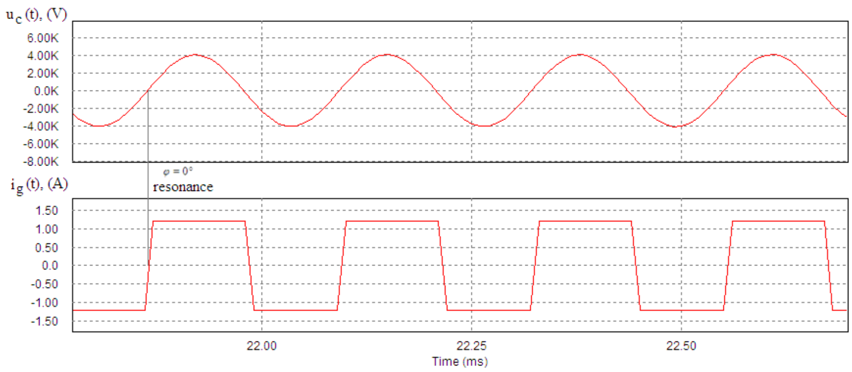
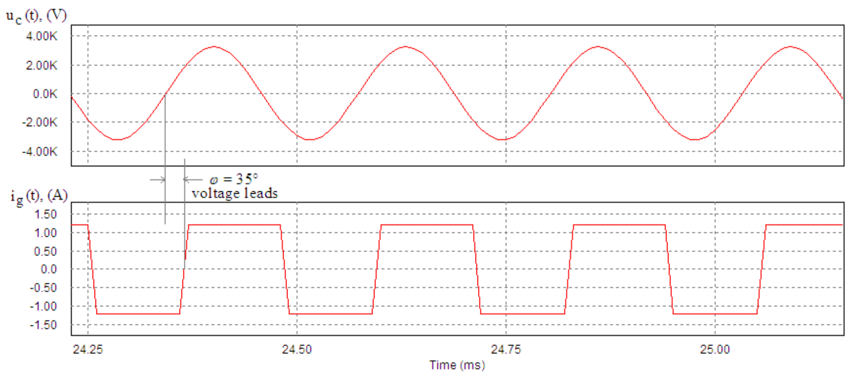


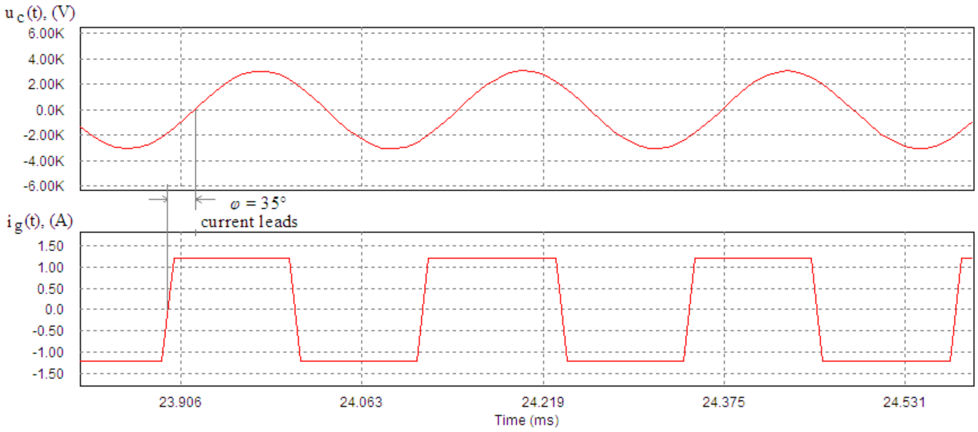
Figure 2. *Serial RL parallel C circuit for simulations*



a.)



b.)



c.)

Figure 3. *Waveform of the voltage and current in the circuit: a.) waveforms of the voltage and current in the circuit when the switching frequency is the same as the damping frequency, b.) waveforms of the voltage and current in the circuit for switching frequency smaller than the damping frequency c.) waveforms of the voltage and current in the circuit for switching frequency bigger than the damping frequency*

From Figure 3a it can be seen that the phase angle between the voltage and the current is zero. In this case, it is said that the circuit works on resonance. In Figure 3b, the voltage leads to the current and it is said that the circuit operates under resonant frequency. In Figure 3c, the current leads to the voltage and it is said that the circuit operates above resonant frequency.

Table 1 gives the data for the effective values of the voltage and current and the power for the three cases from Figure 3.

Table 1 *Data for the effective values of the voltage and current and the power for the three cases from Figure 3*

U_C (V)	I_g (A)	P (W)	φ (°)	
2250	1.2	1920	35	under resonant frequency
2883	1.2	3081	0	on resonance
2166	1.2	1722	35	above resonant frequency

From Table 1 it can be concluded that the power of the consumer is biggest when the switching frequency is the same as the resonant (damping) frequency.

5. Conclusion

In this paper, the serial *RL* parallel *C* circuit was analyzed. First, the second order differential equation that describes this circuit and its solution was represented. After that, simulations of the circuit with given parameters were performed. From the simulations it can be concluded that by changing the phase angle between voltage and current, the power in the circuit can be controlled. At frequencies below the resonant, current of the inductor is greater than the capacitor current. Also, on the resonant frequency the currents on the inductor and the capacitor are the same, and the power of the consumer is biggest. For frequencies above the resonant, the current of the capacitor is greater than the inductor current. This is actually the main purpose why in this paper a solution of the starting differential equation is presented. In fact, the solution of the differential equation for the resonant periodic case describes the operation of the serial *RL* parallel *C* circuit in resonant mode.

References

[1] Williams W. B., (2006). Principles and Elements of Power Electronics, University of Strathclyde, Glasgow.
[2] Shepherd W., Zhang L., (2004). Power Converter Circuits, Marcel Dekker, Ch. 15.
[3] Rao V.K, Rama S.K., Rao M.G., (2010). Pulse and Digital Circuits, Pearson.
[4] Kumar A.A., (2014). Pulse and Digital Circuits, PHI.
[5] Chasnov J. R., (2009). Introduction to Differential Equations: The Hong Kong University of Science and Technology, Adapted for Coursera: Differential Equations for Engineers.
[6] James G., (1996). Modern Engineering Mathematics: Pearson Education Limited, book.
[7] Trench W. F., (2001). Elementary differential equations: Cole Thomson Learning, book.
[8] Stefanov G., Karadzinov Lj., Zlatanovska B., (2011). Mathematical Calculation of H–Bridge IGBT Power Converter, Comptes rendus de l´Academie bulgare des Sciences, Volume 64, Issue No6, pp.897–904.
[9] PowerSim Software, http://www.powersim.com/

Goce Stefanov
University of Goce Delcev Stip
Electrical Engineering Faculty
22 October bb, 2420 Radovis
R. N. Macedonia
goce.stefanov@ugd.edu.mk

Jasmina Veta Buralieva
University of Goce Delcev Stip
Faculty of Computer Science
Krste Misirkov“ no. 10-A, 2000 Stip
R. N. Macedonia
jasmina.buralieva@ugd.edu.mk

Maja Kukuseva Paneva
University of Goce Delcev Stip
Electrical Engineering Faculty
22 October bb, 2420 Radovis
R. N. Macedonia
maja.kukuseva@ugd.edu.mk

Biljana Citkuseva Dimitrovska
University of Goce Delcev Stip
Electrical Engineering Faculty
22 October bb, 2420 Radovis
R. N. Macedonia
biljana.citkuseva@ugd.edu.mk